Eur. Phys. J. B **61**, 335–342 (2008)
DOI: 10.1140/epjb/e2008-00081-5

THE EUROPEAN PHYSICAL JOURNAL B

Adaptive synchronization of weighted complex dynamical networks through pinning

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Received 18 June 2007 / Received in final form 23 October 2007
Published online 29 February 2008 –

Abstract. This paper considers the problem of controlling weighted complex dynamical networks by applying adaptive control to a fraction of network nodes. We investigate the local and global synchronization of the controlled dynamical network through the construction of a master stability function and a Lyapunov function. Analytical results show that a certain number of nodes can be controlled by using adaptive pinning to ensure the synchronization of the entire network. We present numerical simulations to verify the effectiveness of the proposed scheme. In comparison with feedback pinning, the proposed pinning control scheme is robust when tested by noise, different weighting and coupling structures, and time delays.

PACS. 05.45.-a Nonlinear dynamics and chaos – 05.45.Xt Synchronization; coupled oscillators – 89.75.-k Complex systems

1 Introduction

The problem of synchronization control has been a research subject which has attracted increasing attention since the great practical application of nonlinear systems was recognized. Much investigative attention has been focused on either the chaotic synchronization of a few coupled systems (such as master-slave systems) [1–6] or on the synchronization of large-scale networks with regular topological structures [7–10]. More and more studies show that control of the complex dynamics which take place on complex networks – such as the contemporaneous beats of the heart cells [11], or the rhythmic applause in a concert-hall [12] – is an issue of primary importance.

When applied, current concepts have great difficulty in regulating the behaviour of complex dynamical networks. On one hand, various complex networks in nature and society consist of a large set of interconnected nodes, but in which each node can be a dynamical subsystem. All these coupled subsystems lead to topological and statistical similarities [13] – such as small-world effects [14] and scale-free features [15] – which are completely removed from traditional concepts. On the other hand, controlling each node so that each follows a desired synchronous evolution is not always an available or reasonable job (such as in a military hierarchy or in Internet broadcast applications). For example, the synchronous beats of heart cells are regulated by the activity of pacemaker cells situated at the sinoatrial node [11]. Recalling the distributed nature of complex networks, it is feasible to control them by acting locally on certain nodes, and then through coupling between nodes, achieving synchronization of the entire network. Thus, pinning control has been proposed to provide an insight into regulatory mechanisms for controlling networks of coupled dynamical systems [16–18].

The general idea of pinning control is to apply localized feedback to a small fraction of network nodes to achieve control over a given synchronous evolution. In considering the heterogeneity effects of complex networks, there exist different combination styles by selecting different nodes. Li et al. presented two typical selection strategies [19]: (1) "Random pinning", wherein the pinned nodes are randomly selected with uniform probability amongst all the nodes and (2) "Selective pinning" wherein the controlled nodes are first sorted according to some property such as degree, weight, or centrality, and are then chosen sequentially. Interestingly, if the coupling strength is large enough, the coupled dynamical network can achieve synchronization by pinning only one feedback controller [20].

Many researchers have investigated the controllability and stability problems that exist in pinning control [21–23]. In practice, networks do not obey precise state equations without any noise or uncertainties, and feedback gain cannot be arbitrarily large. As well, the topologies of complex networks are often associated with a large heterogeneity in the capacity and intensity of

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the interconnections [24–26]. Most of the early works on pinning control have focused on symmetric unweighted networks [18,19,21,22]. Therefore, the following question emerges: can one solve the above problems by modifying the feedback? In this paper, we present a novel pinning control scheme – replacing local feedback control by a simple adaptive control – to synchronize a given weighted complex dynamical network.

The rest of this paper is organized as follows: a weighted dynamical network model and its adaptive pinning control scheme are presented in Section 2. In Section 3, we investigate the local stability and global stability of the weighted network by using a master stability function (MSF) [27] and a Lyapunov function approach, respectively. In Section 4, numerical simulations are provided to verify the effectiveness of the proposed scheme, and we further discuss the robustness of the adaptive controller. Conclusions are finally given in Section 5.

2 Problem formulation

The dynamics of a general weighted network of N coupled identical oscillators is described by

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} G_{ij} h(x_j), \quad i = 1, 2, \ldots, N, \tag{1}$$

where x_i is the n-dimensional vector of dynamical variables of the ith node, $f(\cdot) \in R^n$ describes the dynamics of each individual oscillator, $h(\cdot) \in R^n$ is the output function, σ is the overall coupling strength, and $G = (G_{ij}) \in R^{N\times N}$ is the coupling matrix, which describes the topology and weights. The entry G_{ij} is zero if there is no connection between node i and node $j \neq i$, but is negative if there is a direct influence from node j, where $|G_{ij}|$ gives a measure of the strength of the interaction, and $G_{ii} = -\sum_{i=1, i\neq j}^{N} G_{ij}$, $i = 1, 2, \ldots, N$, which ensures complete synchronization of the nodes in network (1).

Hereafter, the network is assumed to be connected without any isolated clusters, i.e., G is irreducible. In this work, we focus on a class of weighted networks where G is diagonalizable and has real eigenvalues. In particular, G can be written as $G = DL$, where D is a nonsingular diagonal matrix, and L is a symmetric, zero row-sum, semi-positive definite matrix [28]. For example, if the network (1) is undirected and unweighted, then D is chosen as an identity matrix and L is the Laplacian matrix of the network. Another example is a weighted network with weights

$$G_{ij} = L_{ij}/k_i^{\beta},$$

where k_i is the degree of node i, and β is a tunable parameter [29].

The nodes are said to achieve complete synchronization if

$$\lim_{t\to\infty} ||x_i(t) - s(t)|| = 0, \; i = 1, 2, \ldots, N, \tag{2}$$

where the notation $||\cdot||$ stands for the Euclidean vector norm, and the synchronous state $s(t) \in R^n$ – usually called synchronization manifold - is a solution for an individual node, i.e.,

$$\dot{s}(t) = f(s). \tag{3}$$

Our goal is to achieve complete synchronization by using an adaptive pinning strategy. We apply the adaptive control on a small fraction δ $(0 < \delta \ll 1)$ of the nodes in the network (1). Without loss of generality, let the first l nodes be controlled, as identified by the set $\mathcal{C} = \{c_1, c_2, \ldots, c_l\}$, where $l = \lfloor \delta N \rfloor$ is the integer part of the real number δN. Thus, the controlled network can be described as

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} G_{ij} h(x_j) + u_i, \; i = 1, 2, \ldots, N, \tag{4}$$

Here, the control input is generated by a simple adaptive feedback law:

$$u_i = -\sigma k_i B_i (h(x_i) - h(s)), \quad i = 1, 2, \ldots, N, \tag{5}$$

where B_i is a binary vector. $B_i = 1$ if node i is controlled, otherwise $B_i = 0$, and the adaptive gains k_i satisfy:

$$\dot{k}_i = d_i ||h(x_i) - h(s)||^2, \quad i = 1, 2, \ldots, N \tag{6}$$

where $d_i > 0$ and initial values $k_i(0) > 0$(to guarantee negative feedback).

Note that the control input $u_i(t)$ has a direct influence only on the nodes belonging to the set $\mathcal{C}$. As commonly shown in pinning control schemes, such nodes in set $\mathcal{C}$ play the role of leading the others toward the desired evolution $s(t)$.

To achieve this goal, one should decide the number of controlled nodes l. The decision is influenced by synchronizability of the network (1) and by selection strategies. A better synchronizability will probably lead to less cost in terms of control. Also, it should be noted that applying different strategies for a given network will result in different values of l. A smaller l also means less cost. Assume that the set $\mathcal{C}$ contains l nodes, where l is a fixed number. Then there are $\binom{N}{l}$ different possible ways in which the nodes may be chosen. As has been shown in references [8,27,30–32], the coupling matrix G and the coupling strength σ directly affect the synchronizability of the network; these parameters also affect the detailed selection, as follows from the above description. Thus it is not easy to decide the selection strategy. Without further explanation, for the rest of this paper, we assume G and σ to be known.

3 Stability analysis

This section focuses on the stability analysis of the controlled weighted network (4). By using a MSF and a Lyapunov function approach, we derive the theoretical results for local and global stability of the synchronized state, respectively.

3.1 Local stability analysis

If we define matrix $\widetilde{G}(t) = (\widetilde{G}_{ij}(t)) \in R^{N\times N}$ as

$$\widetilde{G} = \begin{pmatrix} G_{11}+k_1B_1 & G_{12} & \dots & G_{1N} \\ G_{21} & G_{21}+k_2B_2 & \dots & G_{2N} \\ \vdots & \ddots & & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN}+k_NB_N \end{pmatrix},$$

then the pinned network (4) can be written as

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} \widetilde{G}_{ij} h(x_j),\ \ i = 1,2,\dots,N, \qquad (7)$$

where $B_i = 1$ for $i = 1,2,\dots,l$ and $B_i = 0$ for all $i > l$.

Let $K(t) = \mathrm{diag}\{k_1(t)B_1, k_2(t)B_2, \dots, k_N(t)B_N\}$ and $K = D\hat{K}$, where $\hat{K}$ is a diagonal matrix. Therefore, the spectrum of eigenvalues of matrix $\widetilde{G}$ is equal to the spectrum of a symmetric matrix $\hat{G}(t)$, which is defined as

$$\hat{G}(t) = D^{\frac{1}{2}}(L + \hat{K}(t))D^{\frac{1}{2}}. \qquad (8)$$

In other words, the matrix $\widetilde{G}$ can be diagonalized with $\rho(\widetilde{G}) = \rho(\hat{G})$, where $\rho(\cdot)$ denotes the set of eigenvalues of the corresponding matrix. Note that L is an irreducible matrix such that $L \geq 0$, and diagonal matrix $\hat{K} \geq 0$, we easily derive $\hat{G} > 0$ since $L + \hat{K}$ is an irreducible and weakly diagonally dominant matrix whose off-diagonal entries are all negative, where the expression $L > 0$ (or $\geq$, $<$, $\leq$) means that the symmetric matrix L is positive (or semi-positive, negative, semi-positive) definite. In particular, let $\rho(\widetilde{G}) = \{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_N\}$ and

$$0 < \widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \dots \leq \widetilde{\lambda}_N.$$

As has been shown in ref.[21], $\widetilde{G}$ is an asymmetric matrix whose row-sum is no longer equal to zero. We therefore need to extend the master stability function approach to define and assess the controlled network (4). We consider an extended network of $N+1$ dynamical nodes $y_i(t)$, where $y_i(t) = x_i(t)$ for $i = 1,2,\dots,N$ and $y_{N+1}(t) = s(t)$. That is to say, the desired synchronization manifold is given by an extra virtual node added to the original network. Thus, we rewrite equation (7) as

$$\dot{y}_i = f(y_i) - \sigma \sum_{j=1}^{N} \mathcal{G}_{ij} h(y_j),\ \ i = 1,2,\dots,N, \qquad (9)$$

where $\mathcal{G} = (\mathcal{G}_{ij})$ is an $(N+1)\times(N+1)$ square matrix such that

$$\mathcal{G} = \begin{pmatrix} \widetilde{G} & -B \\ 0 & 0 \end{pmatrix} \qquad (10)$$

where $B = (k_1B_1, k_2B_2, \dots, k_NB_N)^{\mathrm{T}}$. It is easily found that zero is one of the eigenvalues of the matrix $\mathcal{G}$. By defining $z_i \in R^n$ as the corresponding eigenvector of $\widetilde{\lambda}_i$, we have $\widetilde{G}z_i = \widetilde{\lambda}_i z_i$. Substituting equation (10) into the above equation yields

$$\mathcal{G}\begin{pmatrix} z_i \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{G}z_i \\ 0 \end{pmatrix} = \widetilde{\lambda}_i \begin{pmatrix} z_i \\ 0 \end{pmatrix}. \qquad (11)$$

Equation (11) shows that $\rho(\mathcal{G}) = \rho(\widetilde{G}) \cup \{0\}$. Then we can directly apply MSF approach to network (9).

To study the stability of the synchronized state, we need the variational equation derived from equation (9):

$$\dot{\xi} = (I_N \otimes \mathbf{D}f - \sigma\widetilde{G} \otimes \mathbf{D}h)\xi \qquad (12)$$

where I_N is an $N\times N$ identity matrix, $\otimes$ is the Kronecker product notation, and $\mathbf{D}f$ and $\mathbf{D}h$ are the Jacobian matrices of functions $f(\cdot)$ and $h(\cdot)$ evaluated on the synchronization manifold $s(t)$, respectively. We assume that the evaluation of the Jacobian of equation (9) leads to a constant matrix on the synchronization manifold, which is true for any linear coupling scheme.

Recalling the rules for block matrix manipulations using the Kronecker product [33] – namely that (1) $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$; (2) $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$, where A, B, C and D are matrices with proper dimensions – the variational equations of equation (9) can be diagonalized by N uncoupled equations

$$\dot{\xi}_i = (\mathbf{D}f - \alpha\mathbf{D}h)\xi_i,\ \ \ i = 1,2,\dots,N \qquad (13)$$

where $\alpha(t) = \sigma\widetilde{\lambda}_i(t)$. The largest Lyapunov exponent $\Gamma(\alpha)$ of this equation can be regarded as a master stability function which determines the linear stability of the synchronized state. The synchronized state is therefore stable if $\Gamma(\sigma\widetilde{\lambda}_i) < 0$ for $i = 1,2,\dots,N$. In other words, the dynamical network (7) achieves locally asymptotically stable if and only if,

$$\sigma\widetilde{\lambda}_i(t, t > T) \in \mathcal{S},\ \ i = 1,2,\dots,N, \qquad (14)$$

where T is a sufficiently large constant, and $\mathcal{S}$ is the synchronized region, governed by $\mathbf{D}f$, $\mathbf{D}h$ and $s(t)$. To distinguish the synchronizability problem of an uncontrolled network (1), the synchronized region $\mathcal{S}$ here is assumed to be an unbounded region, i.e., $\mathcal{S} = \{\alpha | \alpha \in [\alpha_{\min}, \infty)\}$.

According to equation (6), the adaptive gains $k_i(t)$ for all $i = 1,2,\dots,N$ evolve in proportion to the error between the output of node i and the synchronization manifold. As a result, $\widetilde{G}$ is a time-varying matrix and $\widetilde{\lambda}_i(t)$ also varies with time. An important result is that the $k_i(t)$ values are monotonically increasing functions, since $\dot{k}_i(t) \geq 0$. For any $t_1 > t_2$, $\hat{G}(t_1) - \hat{G}(t_2) = K(t_1) - K(t_2) \geq 0$, which indicates that every eigenvalue $\widetilde{\lambda}_i(t)$ evolves with time as a monotonically increasing function. If there exists a fixed diagonal matrix $K^c \in R^{N\times N}$ such that

$$\sigma\mu_1(G + K^c) \geq \alpha_{\min}. \qquad (15)$$

Equation (14) will definitely hold as time evolves, where $\mu_1(\cdot)$ is the smallest eigenvalue of the corresponding matrix. If the control law of equation (6) is replaced by a local injection method, K becomes the constant matrix K^c and equation (15) is equivalent to that for feedback pinning control [19]. In other words, there is no theoretical differences in linear stability between the adaptive pinning (4)–(6) and the feedback pinning method.

3.2 Global stability analysis

It is well-known that the conditions for global stability need to be much more rigorous in comparison to those of linear stability. For instance, the largest controllability of the controlled network (4) is achieved by applying the proposed strategy (5), (6) to each node, i.e., $l = N$. Then the dynamical network (4) always achieves local synchronization about the manifold $s(t)$ since K^c is a positive definite matrix. But for global stability, we cannot derive similar results due to the complicated coupling relations. Here we consider a simplified model, where the output function is linear with respect to the states of nodes, in which the coupling matrix has full rank, i.e., $h(x_i) = x_i$.

Let errors $e_i(t) = x_i(t) - s(t)$, then the state equation can be given by,

$$\dot{e}_i = f(x_i) - f(s) - \sigma \sum_{j=1}^{N} G_{ij} e_j - \sigma k_i B_i e_i \qquad (16)$$

for all $i = 1, 2, \ldots, N$.

It is clear that the dynamical system (4) achieves global asymptotical synchronization if the errors in equation (16) damp out. The following shows how to derive a sufficient condition for global synchronization by using the Lyapunov function approach. Selecting a Lyapunov function as

$$V(e_1, e_2, \ldots, e_N) = \frac{1}{2} \sum_{i=1}^{N} \left(e_i^{\mathrm{T}} e_i + \frac{\sigma}{d_i} (k_i B_i - \gamma)^2 \right) \qquad (17)$$

where γ is arbitrarily chosen.

Then the derivative of $V(e_1, e_2, \ldots, e_N)$ along with the synchronization manifold $s(t)$ is

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} e_i^{\mathrm{T}} \dot{e}_i + \sigma \sum_{i=1}^{N} B_i (k_i - \gamma) e_i^{\mathrm{T}} e_i \\
&= \sum_{i=1}^{N} e_i^{\mathrm{T}} (f(x_i) - f(s) - \sigma \sum_{j=1}^{N} \widetilde{G}^c_{ij} e_j) \\
&= \sum_{i=1}^{N} e_i^{\mathrm{T}} [(f(x_i) - f(s) - \Delta e_i) - (\sigma \sum_{j=1}^{N} \widetilde{G}^c_{ij} e_j - \Delta e_i)] \\
&= \sum_{i=1}^{N} e_i^{\mathrm{T}} (f(x_i) - f(s) - \Delta e_i) \\
&\quad - e^{\mathrm{T}} (\sigma \widetilde{G}^c \otimes I_n - I_N \otimes \Delta) e
\end{aligned}$$

where $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$ is a diagonal matrix, $e(t) = (e_i^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t))^{\mathrm{T}}$, and $\widetilde{G}^c = (\widetilde{G}^c_{ij})$ is an $N \times N$ constant matrix such that

$$\widetilde{G}^c = G + K^c$$
$$= \begin{pmatrix} G_{11} + \gamma B_1 & G_{12} & \ldots & G_{1N} \\ G_{21} & G_{21} + \gamma B_2 & \ldots & G_{2N} \\ \vdots & \ddots & & \vdots \\ G_{N1} & G_{N2} & \ldots & G_{NN} + \gamma B_N \end{pmatrix}$$

It follows from the description in Section 3.1 that $\mu_1(\widetilde{G}^c) > 0$, where $\mu_1(\cdot)$ is the smallest eigenvalue of the corresponding matrix.

In order to guarantee the negativeness of the derivative of $V(e_1, e_2, \ldots, e_N)$, we assume that

$$\sigma \mu_1(\widetilde{G}^c) - \Delta_i > 0, \; i = 1, 2, \ldots, N, \qquad (18)$$

and for any vectors x, $y \in R^n$,

$$\begin{aligned}
&(x - y)^{\mathrm{T}} (f(x) - f(y) - \Delta(x - y)) \\
&\qquad \leq -\eta (x - y)^{\mathrm{T}} (x - y), \qquad (19)
\end{aligned}$$

where $\eta > 0$. Recalling the properties of Kronecker products again, we have $\mu_1(\sigma \widetilde{G}^c \otimes I_n - I_N \otimes \Delta) = \mu_1(\sigma \widetilde{G}^c) - \max\{\Delta_i, i = 1, 2, \ldots, n\}$. As a consequence of assumption (18) and proposition 1 (see Appendix for details), we can deduce $e^{\mathrm{T}}(\sigma \widetilde{G}^c - \Delta) e \geq 0$. Therefore, we have $\dot{V} \leq -\eta e^{\mathrm{T}} e$. It is apparent that $\dot{V} = 0$ if and only if $e(t) = 0$. According to the Lyapunov stability theorem, the dynamical system (16) is asymptotically stable.

Note that in the controlled network (7), the synchronous solution $s(t)$ is assumed to be an invariant manifold for linear stability analysis. Here, we suppose that each oscillator $\dot{s} = f(s)$ satisfies a Lipschitz condition, i.e., for two arbitrary different vectors $x_i^1(t)$ and $x_i^2(t)$, the corresponding trajectories should satisfy $||f(x_i^1) - f(x_i^2)|| \leq L_c^f ||x_i^1 - x_i^2||$ for all time t, where L_c^f is a positive constant. Many chaotic oscillators such as the Chua, Lorenz, and Chen systems, satisfy such a condition. If we select $\Delta_i = L_c^f$ for all $i = 1, 2, \ldots, n$, then the condition of global stability of system (7) can be rewritten as

$$\sigma \mu_1(\widetilde{G}^c) > L_c^f. \qquad (20)$$

The same result for feedback pinning has been derived in reference [19].

3.3 Further discussions on stability

In the two subsections above, the stability of the controlled network (4) is governed by the smallest eigenvalue of the matrix $\widetilde{G}^c$, where $K^c \geq 0$ is a diagonal matrix whose dimension is equal to the number of pinned nodes, l. In linear stability analysis, the synchronized region $\mathcal{S}$ is assumed to be unbounded. If $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ with a fixed constant $\alpha_{\max}$, the ratio $r = \mu_N(\widetilde{G}^c)/\mu_1(\widetilde{G}^c)$ will measure the synchronizability of the network, where $\mu_N(\widetilde{G}^c)$

is the largest eigenvalue of the matrix $\widetilde{G}^c$. For simplicity, let $K^c = k^c \mathrm{diag}\{1, 1, \ldots, 1, 0, \ldots, 0\}$ and k^c be positive constant. The ratio r will not increase monotonically with feedback gain k^c [21]. However, for an unbounded region, the measure $\mu_1(\widetilde{G}^c)$ – the so-called controllability – increases monotonically with k^c. This result provides evidence that the same analytical results from feedback pinning can be applied to adaptive pinning. For a given complex dynamical network (7), the controllability, $\mu_1(\widetilde{G}^c)$, is affected by the gain k^c, the number of pinned nodes l, and the method in which l different nodes are selected from N nodes. The smallest eigenvalue $\mu_1(\widetilde{G}^c)$ increases monotonically with k^c. The value of l plays a key role; a greater l leads to most probably to a larger $\mu_1(\widetilde{G}^c)$. An extreme case is when synchronize the complex dynamical network (4) by introducing a single adaptive controller (the simplest case). An obvious conclusion is that $\mu_1(\widetilde{G}^c) > \epsilon > 0$ if $\epsilon > \alpha_{\min}/\sigma$ from equation (15). As a result, we can select any one node in the network to introduce an adaptive control law (5), (6) as long as the coupling strength σ is large enough. Another extreme case is to apply the control law to every node (the scheme resulting in the highest controllability). Then $B_i = 1$ for all $i = 1, 2, \ldots, N$. If the Lyapunov function is still selected the same as equation (17), we can easily deduce that the dynamical network (4), no matter how it is initialized, will achieve global synchronization about $s(t)$. The result tells us that we can handle any network topology – as defined in Section 2 – by using the proposed control scheme. From the coupling strength point of view, a large number of controlled nodes l will reduce the critical strength of the synchronizability for a weighted complex dynamical network. For the general case ($1 < l < N$), determining l is not easy, since various combinations of selecting l nodes will probably lead to different values $\mu_1(\widetilde{G}^c)$. Generally speaking, the selection procedure usually needs large-scale computation and data processing. The related optimization problem will be left as our future work.

4 Numerical results and discussions

In this section, we present numerical simulations to show the effectiveness of the proposed adaptive control scheme and give a brief discussion on the robustness of the presented control scheme. We consider a random, scale-free network (here, we use random pseudofractal network, RPN, see Refs. [34–36]) as the numerical model. The model can be described as follows: the growth starts from a single edge with two nodes. At each time step, a new node with two edges is added to every existing node, where the new edges are attached to both ends of the corresponding edge. Repeating this rule will produce a desirable RPN. The RPN exhibits the scale-free property and the small-world effect simultaneously, with a power-law exponent $\gamma = 3$. In the following simulations, we take $G_{ij} = L_{ij}/k_{ij}^{\beta}$ and the output function $h(x) = x$, where β is a tunable parameter, L is the Laplacian matrix of the RPN. We use

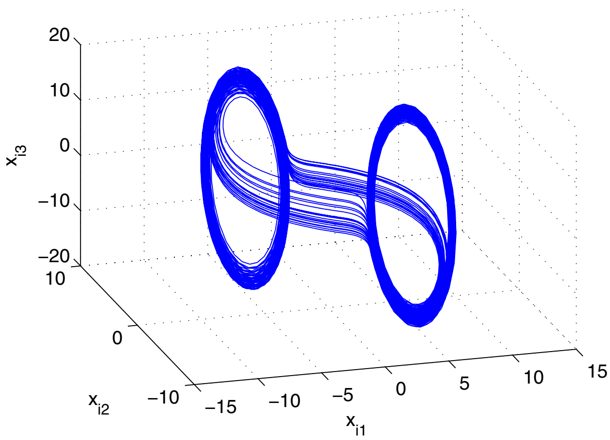

Fig. 1. Chaotic attractor (Chua oscillator (21)) with parameters given in equation (22), where $x_{ij}(t)$ is the jth state of the node i, and the initial vector $x_i(0) = (1.5, -4.4, 0.15)^{\mathrm{T}}$.

a Chua chaotic oscillator [37,38] as a dynamical node in the RPN, which is described in dimensionless form by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} a(x_{i2} - x_{i1} - f_1(x_{i1})) \\ b(x_{i1} - x_{i2}) + cx_{i3} \\ -dx_{i2} \end{pmatrix} \tag{21}$$

where $f_1(x_{i1}) = m_0 x_{i1} + \frac{1}{2}(m_1 - m_0)(|x_{i1} + 1| - |x_{i1} - 1|)$. Here, we chose the parameters as:

$$\begin{array}{l} a = 7, \quad b = \frac{7}{20}, \qquad d = 7, \\ c = \frac{1}{2}, \; m_0 = -\frac{1}{7}, \; m_1 = -\frac{40}{7}. \end{array} \tag{22}$$

A single Chua oscillator is shown in Figure 1. The adaptive pinning control scheme (5), (6) is then introduced to control the dynamical network. Figure 2 shows the error trajectories of all nodes when pinning only one node, where $e_{ij}(t)$ is the jth state error of node i, $\beta = 0$, indicating the coupling matrix G is an unweighted symmetric matrix. Figure 3 shows the error trajectories of all nodes achieved by pinning two nodes with $\beta = 1$. The above two simulations show that a dynamical network – weighted or unweighted, symmetric or asymmetric – can be stabilized through adaptive pinning control.

In comparison with Section 3, these analytical results – including the local stability and global stability – are the same as when using pinning control through local injection. A question naturally arises: why replace injection with an adaptive control law? An obvious answer is that adaptive pinning ensures that synchronization of the complex dynamical network (4) will be reached automatically without any prior knowledge of feedback gain. Also, in using a pinning process, we can find a lower bound $k_{\min}$ for different initial values $k_i(0)$ and d_i, where i is the selected node.

Figure 4 shows the evolution of the adaptive gains $k_i(t)$. However, in feedback pinning, not all negative feedback control schemes are able to guarantee the synchronization of the whole controlled network (4). We therefore have to select sufficiently large feedback gains k_i for

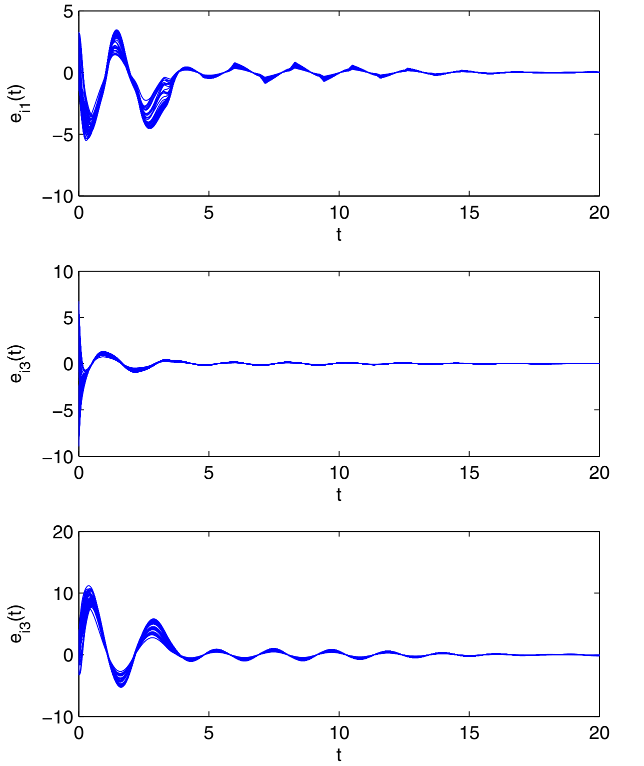

Fig. 2. Fifty Chua oscillators coupled by an unweighted RPN with one adaptive controller, where gains $k_1(0) = 1$, $d_1 = 1$, $\beta = 0$, and coupling strength $\sigma = 5$.

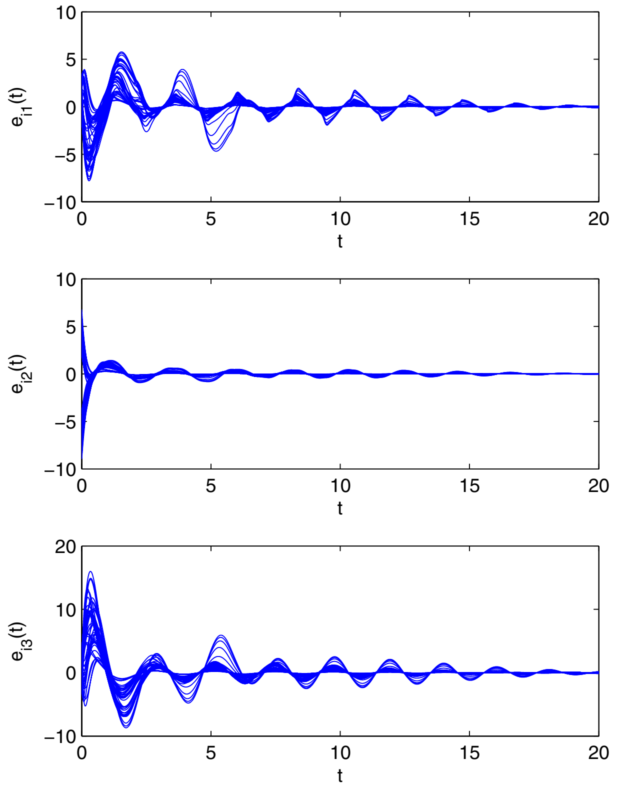

Fig. 3. Fifty Chua oscillators coupled by a weighted RPN with two adaptive controllers, where gains $k_1(0) = k_2(0) = 1$, $d_1 = d_2 = 1$, $\beta = 1$, and coupling strength $\sigma = 5$.

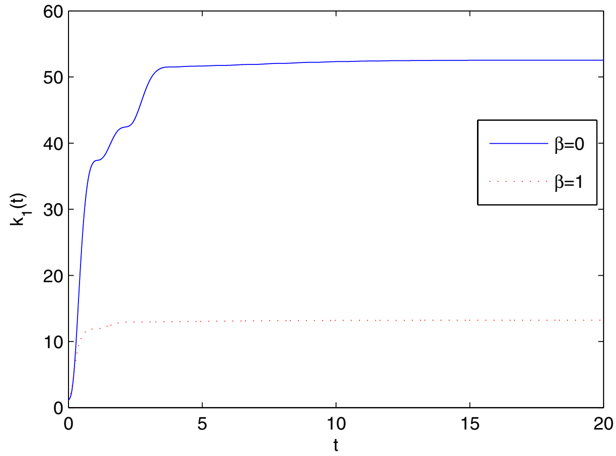

Fig. 4. (Color on line) Evolution of adaptive gains $k_1(t)$ in unweighted and weighted RPNs with $\beta = 0$ (blue curve) and $\beta = 1$ (red dashed), where the parameters including the network size, coupling matrix, control strategies are all the same as in Figures 2 and 3. As for Figure 3, there are two nodes to be controlled. The adaptive gains $k_1(t)$ and $k_2(t)$ are almost identical, so only the evolution of $k_1(t)$ is shown.

$i = 1, 2, \ldots, l$. A common problem is the use an overloaded control strength, which is obtained only through repeated examinations. Another advantage of the presented control scheme is its strong robustness against noise. In order to demonstrate, we introduce the quantity $\mathcal{Q}(t) = \sqrt{(\sum_{i=1}^{N} ||x_i(t) - s(t)||^2)/N}$, which is used to measure the quality of the pinning process. We also assume that there exists some disturbance to the process of measuring the synchronization manifold at a certain time. We then adopt feedback pinning and adaptive pinning to control the network, where the network parameters are the same as above. Note that we still select the same single node to control the network. Figure 5 shows the comparison of the adaptive pinning and feedback pinning qualities $\mathcal{Q}(t)$. From the figure, one can see that it is very hard to tell which control scheme is better in the absence of noise ($t < 10$), while the control performance of adaptive control is much better than that of feedback control in the presence of noise ($t > 10$).

In previous discussions, the weighted complex dynamical network is assumed to be given beforehand without any unknown parameters. In practice, however, such an assumption is unrealistic because of the complexity of these networks, including the large-scale nodes, unknown or uncertain coupling relations and topology, and time-delays. In this situation, feedback pinning has difficulty in synchronizing these uncertain, complex dynamical networks. However, a concise result is that the adaptive pinning proposed in this paper can always obtain stability in such uncertain systems described as

$$\dot{x}_i = f(x_i) - \sigma(t) \sum_{j=1}^{N} G_{ij}(t) x_j(t - \tau(t)), \ i = 1, 2, \ldots, N, \tag{23}$$

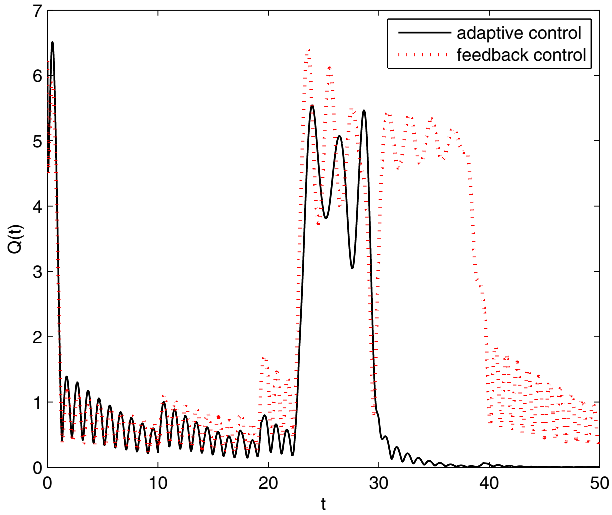

Fig. 5. (Color online) Comparison of adaptive pinning and feedback pinning qualities $\mathcal{Q}(t)$, where $\sigma = 5$, adaptive gain $k_1(0) = 1$, $d_1 = 1$ and feedback gain $k = 20$. At $t = 10$, a white noise is introduced to each node, and the control laws are kept the same. It is obvious that the system under adaptive control has a much smaller disturbance error and a faster speed of convergence than the one under feedback control.

if $l = N$, where the coupling strength $\sigma(t)$, coupling matrix $G(t)$ and time delay $\tau(t)$ can be any bounded uncertainties. In other words, there must exist a certain constant l such that $1 \leq l \leq N$ to ensure the stability of the weighted complex dynamical network (16) since adaptive pinning control supplies an upper bound that is larger than the minimum control depends, independent of the network structure.

Remark. The detailed proof described above is similar to that of Section 3.2 if the Lyapunov functional is selected as

$$V = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}} e_i + \frac{1}{2}\sum_{i=1}^{N} \frac{\overline{\sigma}}{d_i}(k_i B_i - \gamma)^2 + \frac{1}{2}\sum_{i=1}^{N}\left(\int_t^{t+\tau} e_i(t-\tau)^{\mathrm{T}} e_i(t-\tau)dt\right) \tag{24}$$

where $\sigma(t) \leq \overline{\sigma}$, $\overline{\sigma}$ is a positive constant, and $\tau(t) = \tau$ is a time-varying function such that $|\dot{\tau}(t)| < 1$.

5 Conclusion

In this paper, we have investigated synchronization problems for a weighted complex dynamical network via pinning control. The general strategy is to apply an adaptive control scheme to a small fraction of the network nodes. By using the master stability function and the Lyapunov function approach, we deduce theoretical results for local and global stabilization of the synchronization manifold, respectively. The analytical results show that for an unbounded synchronized region $\mathcal{S}$, the smallest eigenvalue of matrix $G + K^c$ determines the synchronization of the weighted complex dynamical network (4) (both local and global synchronization), where the diagonal matrix K^c supplies the number of controlled nodes, the feedback strength, and the set of selected nodes. By seeking an appropriate K^c, we are able to achieve our goal. All these results are the same as those for pinning control using local feedback. Numerical simulations show that in comparison with feedback pinning, the proposed control strategy has strong robustness against noise. In particular, adaptive pinning control is still effective on a weighted uncertain complex dynamical network with time-varying delays, as long as a sufficient number of nodes are controlled. When $l = N$, this control strategy can always ensure the synchronization of the controlled networks, independent of any knowledge of the network structure, coupling relations, or of the strength.

The authors thank the referees and the editor for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (No. 60736021), the Key Technologies R&D Program of Zhejiang Province (No. 2006C11066), the Specialized Research Fund for the Doctoral Program of Higher Education(SRFDP) (No. 20050335020), and the Hi-Tech Research and Development Program (863) of China (No. 2007AA11Z216).

Appendix A

Proposition 1. Given matrix $G \in R^{n\times n}$. If $G = DL$ and $\mu_1(G) > 0$, then for any nonzero vector $x \in R^n$ and $i = 1, 2, \ldots, n$, we have $x^{\mathrm{T}} G x > 0$, where D is a nonsingular diagonal matrix, L is a symmetric matrix, and $\mu_1(G)$ is the smallest eigenvalue of G.

Proof. The following equation holds

$$G = D^{\frac{1}{2}}(D^{\frac{1}{2}} L D^{\frac{1}{2}})D^{-\frac{1}{2}} = D^{\frac{1}{2}}\hat{G}D^{-\frac{1}{2}}, \tag{25}$$

which can be easily derived from $G = DL$, where $\hat{G} = D^{\frac{1}{2}} L D^{\frac{1}{2}}$.

Equation (25) indicates that $\rho(G) = \rho(\hat{G})$ and G can be diagonalized, where $\rho(\cdot)$ is the set of eigenvalues of the corresponding matrix. Then there exist n eigenvalues $\lambda_i \in R^+$ associated with n linearly independent vectors $y_i \in R^n$ for matrix G, $i = 1, 2, \ldots, n$. Moreover,

$$y_i^{\mathrm{T}} G y_j = \lambda_j y_i^{\mathrm{T}} y_j = \begin{cases} 0, & \text{if } i \neq j \\ \lambda_j ||y_i||^2 > 0, & \text{otherwise.} \end{cases} \tag{26}$$

In particular, for any nonzero vector $x \in R^n$, we have

$$x = \sum_{i=1}^{n} \theta_i y_i$$

where $\theta_i \in R$ and $\sum_{i=1}^{n} |\theta_i| \neq 0$.

Furthermore,

$$x^{\mathrm{T}}Gx = \left(\sum_{i=1}^{n}\theta_i y_i^{\mathrm{T}}\right) G \left(\sum_{j=1}^{n}\theta_j y_j\right)$$
$$= \left(\sum_{i=1}^{n}\theta_i y_i^{\mathrm{T}}\right)\left(\sum_{j=1}^{n}\lambda_j\theta_j y_j\right). \quad (27)$$

Recalling equation (26), we have

$$x^{\mathrm{T}}Gx = \sum_{i=1}^{n}\lambda_i\theta_i^2||y_i||^2 > 0. \quad (28)$$

The proof is thus completed.

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